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EFFECTS OF GRADIENT OF POLARIZATION ON STRESS-CONCENTRATION AT A CYLINDRICAL HOLE IN AN ELASTIC DIELECTRIC

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Abstract—The solution of the problem of a cylindrical hole in a field of longitudinal tension is found in the linear theory of elastic dielectrics in which the potential energy density of deformation and polarization depends on the gradient of the polarization as well as on the strain and on the polarization itself. The stress-concentration factor at the surface of the cylindrical hole is found.

1. INTRODUCTION

IN THIS paper a boundary-value problem is solved within the framework of Mindlin's [1] theory of elastic dielectrics with polarization gradient.

In the following section, the field equations and the stress functions [2] analogous to the Papkovitch's functions of classical elasticity are presented. In the third section, the stress functions are applied to solve the boundary-value problem for the stress concentration at a cylindrical hole in a medium of infinite extent subject to a longitudinal tension. It is found that the stress-concentration factor depends upon the radius of the cylindrical hole, three length properties of the material, Poisson's ratio, two Poisson-like ratio and the reciprocal dielectric susceptibility. In the fourth section, the behavior of stress concentration factor is examined by employing the asymptotic representation for the modified Bessel functions. There is a certain range of material properties for which the stress-concentration factor is higher than the constant value 3 obtained by using the classical theory of elasticity.

2. FIELD EQUATIONS AND GENERAL SOLUTION

The field equations for the linear theory of an elastic dielectric with polarization gradient have been presented by Mindlin [1] and are reproduced here for convenience.

Let the body occupy a region V, whose boundary S separates it from a vacuum V'. In the absence of an external body force and an external electric field the "displacement" equations of equilibrium in vector forms are

$$c_{44}\nabla^2\mathbf{u}+(c_{12}+c_{44})\nabla\nabla\cdot\mathbf{u}+d_{44}\nabla^2\mathbf{P}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{P} = 0 \tag{2.1a}$$

$$d_{44}\nabla^2\mathbf{u}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{u}+(b_{44}+b_{77})\nabla^2\mathbf{P}+(b_{12}+b_{44}-b_{77})\nabla\nabla\cdot\mathbf{P}-a\mathbf{P}-\nabla\varphi = 0 \tag{2.1b}$$

$$-\varepsilon_0\nabla^2\varphi+\nabla\cdot\mathbf{P} = 0, \qquad \text{in } V \tag{2.1c}$$

$$\nabla^2\varphi = 0, \qquad \text{in } V'. \tag{2.1d}$$

The boundary conditions for a free surface are

$$\mathbf{n}\,.\,\tau = 0 \tag{2.2a}$$

$$\mathbf{n}\,.\,\mathbf{E} = 0 \tag{2.2b}$$

$$\mathbf{n}\,.\,(-\varepsilon_0[\nabla\varphi]+\mathbf{P}) = 0. \tag{2.2c}$$

In the above equations, $\boldsymbol{\tau}$ is the stress, $\mathbf{E}$ is derivable from the energy density of deformation and polarization W (i.e. $E_{ij} = \partial W/\partial P_{j,i}$), φ is the potential of the Maxwell self-field, $\mathbf{P}$ is the polarization, ε_0 is the permittivity of a vacuum, $\mathbf{n}$ is the unit outward normal, $[\nabla\varphi]$ is the jump in $\nabla\varphi$ across S.

For an isotropic and centrosymmetric material, the energy density W of deformation and polarization is given by

$$W = b_0P_{i,i}+\tfrac{1}{2}aP_iP_i+\tfrac{1}{2}b_{12}P_{i,i}P_{j,j}+\tfrac{1}{2}(b_{44}+b_{77})P_{j,i}P_{j,i}+\tfrac{1}{2}(b_{44}-b_{77})P_{j,i}P_{i,j}$$
$$+\tfrac{1}{2}c_{12}\varepsilon_{ii}\varepsilon_{jj}+c_{44}\varepsilon_{ij}\varepsilon_{ij}+d_{12}P_{i,i}\varepsilon_{jj}+2d_{44}P_{j,i}\varepsilon_{ij}, \tag{2.3}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right) \tag{2.4}$$

in which u_j is the displacement. Then the constitutive relations are

$$-\bar{E}_j = \frac{\partial W}{\partial P_j} = aP_j, \tag{2.5}$$

$$E_{ij} = \frac{\partial W}{\partial P_{j,i}} = b_{12}\delta_{ij}P_{k,k}+(b_{44}+b_{77})P_{j,i}+(b_{44}-b_{77})P_{i,j}+d_{12}\delta_{ij}\varepsilon_{kk}+2d_{44}\varepsilon_{ij}+b_0\delta_{ij} \tag{2.6}$$

$$\tau_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = d_{12}\delta_{ij}P_{k,k}+d_{44}(P_{j,i}+P_{i,j})+c_{12}\delta_{ij}\varepsilon_{kk}+2c_{44}\varepsilon_{ij}. \tag{2.7}$$

Schwartz has shown [2] that any solution $\{\mathbf{u},\mathbf{P},\varphi\}$ of the displacement equations of equilibrium in a region V bounded by a surface S, can be expressed as

$$\mathbf{u} = \mathbf{B}-\frac{\alpha}{2}\nabla(\mathbf{r}\,.\,\mathbf{B}+\mathbf{B}_0)+\frac{c_{44}}{a}k_2(k_2-k_1)\nabla\nabla\,.\,\mathbf{B}-\varepsilon_0k_1\nabla\varphi+\frac{k_2}{a}(1+a\varepsilon_0)$$
$$\times(1-l_1^2\nabla^2)\nabla\varphi-k_2(K-l_2^2\nabla\nabla\,.\,K) \tag{2.8}$$

$$\mathbf{P} = -a^{-1}c_{44}(k_2-k_1)\nabla\nabla\,.\,\mathbf{B}+\varepsilon_0\nabla\varphi-a^{-1}(1+a\varepsilon_0)(1-l_1^2\nabla^2)\nabla\varphi+K-l_2^2\nabla\nabla\,.\,K \tag{2.9}$$

provided that $\mathbf{B}$, B_0, $\mathbf{K}$ and φ satisfy in V, the equations

$$\nabla^2\mathbf{B} = 0 \tag{2.10a}$$

$$\nabla^2B_0 = 0 \tag{2.10b}$$

$$(1-l_2^2\nabla^2)K = 0 \tag{2.10c}$$

$$(1-l_1^2\nabla^2)\nabla^2\varphi = 0 \tag{2.10d}$$

where $\mathbf{r}$ is the position vector, and

$$\alpha = (c_{12}+c_{44})/(c_{12}+2c_{44}) = \tfrac{1}{2}(1-\nu) \tag{2.11a}$$

$$k_1 = (d_{12}+2d_{44})/(c_{12}+2c_{44}) \tag{2.11b}$$

$$k_2 = d_{44}/c_{44} \tag{2.11c}$$

$$l_1^2 = \varepsilon_0(1+a\varepsilon_0)^{-1}[(b_{12}+2b_{44})-K_1(d_{12}+2d_{44})] \tag{2.11d}$$

$$l_2^2 = a^{-1}[(b_{44}+b_{77})-K_2 d_{44}]. \tag{2.11e}$$

Each of the parameters l_1 and l_2 has the dimension of length.

In the cylindrical coordinate system r, θ, z, in a state of plane strain, the vector displacement $\mathbf{u}$ and the vector polarization $\mathbf{P}$ may be written as

$$\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta, \qquad u_z = 0, \tag{2.12}$$

and

$$\mathbf{P} = P_r\mathbf{e}_r + P_\theta\mathbf{e}_\theta, \qquad P_z = 0 \tag{2.13}$$

respectively, where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors positive in the directions r, θ increasing and u_r, u_θ, P_r and P_θ are functions of r and θ.

The components of strain dyadic ε, in plane strain, are

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right), \\ \varepsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}, & \varepsilon_{rz} &= \varepsilon_{\theta z} = \varepsilon_{zz} = 0. \end{aligned} \tag{2.14}$$

According to equations (2.6) and (2.7), the constitutive relations, in cylindrical coordinate, are

$$\tau_{rr} = d_{12}\nabla\cdot\mathbf{P}+2d_{44}\frac{\partial P_r}{\partial r}+c_{12}\nabla\cdot\mathbf{u}+2c_{44}\frac{\partial u_r}{\partial r}, \tag{2.15a}$$

$$\tau_{r\theta} = d_{44}\left(\frac{1}{r}\frac{\partial P_r}{\partial \theta}-\frac{P_\theta}{r}+\frac{\partial P_\theta}{\partial r}\right)+c_{44}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right), \tag{2.15b}$$

$$\tau_{\theta\theta} = d_{12}\nabla\cdot\mathbf{P}+2d_{44}\left(\frac{1}{r}\frac{\partial P_\theta}{\partial \theta}+\frac{P_r}{r}\right)+c_{12}\nabla\cdot\mathbf{u}+2c_{44}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right), \tag{2.15c}$$

$$E_{rr} = b_{12}\nabla\cdot\mathbf{P}+2b_{44}\frac{\partial P_r}{\partial r}+d_{12}\nabla\cdot\mathbf{u}+2d_{44}\frac{\partial u_r}{\partial r}+b_0, \tag{2.15d}$$

$$E_{r\theta} = b_{44}\left(\frac{1}{r}\frac{\partial P_r}{\partial \theta}-\frac{P_\theta}{r}+\frac{\partial P_\theta}{\partial r}\right)+b_{77}\left(\frac{\partial P_\theta}{\partial r}-\frac{1}{r}\frac{\partial P_r}{\partial \theta}+\frac{P_\theta}{r}\right)+d_{44}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right), \tag{2.15e}$$

$$E_{\theta r} = b_{44}\left(\frac{1}{r}\frac{\partial P_r}{\partial \theta}-\frac{P_\theta}{r}+\frac{\partial P_\theta}{\partial r}\right)-b_{77}\left(\frac{\partial P_\theta}{\partial r}-\frac{1}{r}\frac{\partial P_r}{\partial \theta}+\frac{P_\theta}{r}\right)+d_{44}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right). \tag{2.15f}$$

3. SOLUTION BY MEANS OF STRESS FUNCTIONS

In cylindrical coordinates r, θ, z a stress-field of simple tension, T, in the plane of r and θ is given by

$$\tau_{rr} = \tfrac{1}{2}T(1+\cos 2\theta) \tag{3.1a}$$

$$\tau_{\theta\theta} = \tfrac{1}{2}T(1-\cos 2\theta) \tag{3.1b}$$

$$\tau_{r\theta} = -\tfrac{1}{2}T\sin 2\theta. \tag{3.1c}$$

We wish to add a stress field which will produce a free surface at $r = R$ and vanish at infinity. From equations (3.1), the conditions which the additional field must satisfy on $r = R$ are

$$\tau_{rr} = -\tfrac{1}{2}T(1+\cos 2\theta) \tag{3.2a}$$

$$\tau_{r\theta} = \tfrac{1}{2}T\sin 2\theta \tag{3.2b}$$

$$E_{rr} = 0 \tag{3.2c}$$

$$E_{r\theta} = 0 \tag{3.2d}$$

For the addition field, we take the stress functions $\mathbf{B}$, B_0, $\mathbf{K}$ and φ to be of the form

$$\mathbf{B} = B(r,\theta)\mathbf{e}_x, \qquad B_y = B_z = 0 \tag{3.3a}$$

$$B_0 = B_0(r,\theta), \qquad \varphi = \varphi(r,\theta) \tag{3.3b}$$

$$\mathbf{K} = K(r,\theta)\mathbf{e}_x. \tag{3.3c}$$

For B, B_0, K and φ we take

$$B = A_1 r^{-1}\cos\theta \tag{3.4a}$$

$$B_0 = A_3\log r + A_4 r^{-2}\cos 2\theta \tag{3.4b}$$

$$K = A_2 K_1\left(\frac{r}{l_2}\right)\cos\theta \tag{3.4c}$$

$$\varphi = A_5 K_0\left(\frac{r}{l_1}\right) + A_6 K_2\left(\frac{r}{l_1}\right)\cos 2\theta + A_7 r^{-2}\cos 2\theta \tag{3.4d}$$

where $K_0(r/l_1)$, $K_1(r/l_2)$ and $K_2(r/l_1)$ are the modified Bessel functions of the second kind of orders zero, one and two respectively. It may be verified that these functions satisfy equations (2.10) and give displacements and stresses that vanish at infinity.

In terms of stress functions given in equations (3.3), the components of displacement and polarization can be written as

$$u_r = B\cos\theta - \frac{\alpha}{2}\frac{\partial}{\partial r}(rB\cos\theta + B_0) + a^{-1}c_{44}k_2(k_2-k_1)\frac{\partial}{\partial r}(\nabla\cdot\mathbf{B})$$
$$+(k_2a^{-1}+k_2\varepsilon_0-k_1\varepsilon_0)\frac{\partial\varphi}{\partial r} - k_2(a^{-1}+\varepsilon_0)l_1^2\nabla^2\left(\frac{\partial\varphi}{\partial r}\right) - k_2\left(K\cos\theta - l_2^2\frac{\partial}{\partial r}\nabla\cdot\mathbf{K}\right) \tag{3.5a}$$

$$u_\theta = -B\sin\theta - \frac{\alpha}{2}\frac{1}{r}\frac{\partial}{\partial\theta}(rB\cos\theta + B_0) + a^{-1}c_{44}k_2(k_2-k_1)\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla\,.\,\mathbf{B})$$

$$+(k_2a^{-1}+k_2\varepsilon_0-k_1\varepsilon_0)\frac{1}{r}\frac{\partial\varphi}{\partial\theta} - k_2(a^{-1}+\varepsilon_0)l_1^2\nabla^2\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)$$

$$-k_2\left(-K\sin\theta - l_2^2r^{-1}\frac{\partial}{\partial\theta}\nabla\,.\,K\right), \tag{3.5b}$$

$$P_r = -a^{-1}c_{44}(k_2-k_1)\frac{\partial}{\partial r}(\nabla\,.\,\mathbf{B}) + \varepsilon_0\frac{\partial\varphi}{\partial r} - a^{-1}(1+a\varepsilon_0)(1-l_1^2\nabla^2)\frac{\partial\varphi}{\partial r}$$

$$+K\cos\theta - l_2^2\frac{\partial}{\partial r}\nabla\,.\,K, \tag{3.5c}$$

$$P_\theta = -a^{-1}c_{44}(k_2-k_1)\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla\,.\,\mathbf{B}) + \varepsilon_0\frac{1}{r}\frac{\partial u}{\partial\theta} - a^{-1}(1+a\varepsilon_0)(1-l_1^2\nabla^2)\frac{1}{r}\frac{\partial\varphi}{\partial\theta}$$

$$-K\sin\theta - l_2^2\frac{1}{r}\frac{\partial}{\partial\theta}\nabla\,.\,K. \tag{3.5d}$$

Substitution of (3.4) in (3.5) gives

$$u_r = A_1r^{-1} - \frac{\alpha}{2}r^{-1}A_3 + A_5k_1\varepsilon_0l_1^{-1}K_1\left(\frac{r}{l_1}\right) + \left\{A_1\left[\frac{r^{-1}}{2} + 2a^{-1}k_2(k_2-k_1)c_{44}r^{-3}\right]\right.$$

$$+A_2k_2l_2r^{-1}K_2\left(\frac{r}{l_2}\right) + A_4\alpha r^{-3} + A_6k_1\varepsilon_0l_1^{-1}\left[K_1\left(\frac{r}{l_1}\right) + 2l_1r^{-1}K_2\left(\frac{r}{l_1}\right)\right]$$

$$\left.-2A_7r^{-3}(k_2a^{-1}+k_2\varepsilon_0-k_1\varepsilon_0)\right\}\cos 2\theta \tag{3.6a}$$

$$u_\theta = \left\{A_1[\tfrac{1}{2}(\alpha-1)r^{-1} + 2a^{-1}k_2(k_2-k_1)c_{44}r^{-3}] + A_2k_2\left[\frac{1}{2}K_1\left(\frac{r}{l_2}\right) + l_2r^{-1}K_2\left(\frac{r}{l_2}\right)\right]\right.$$

$$\left.+A_4\alpha r^{-3} + 2A_6k_1\varepsilon_0r^{-1}K_2\left(\frac{r}{l_1}\right) - 2A_7r^{-3}(k_2a^{-1}+k_2\varepsilon_0-k_2\varepsilon_0)\right\}\sin 2\theta, \tag{3.6b}$$

$$P_r = -A_5\varepsilon_0l_1^{-1}K_1\left(\frac{r}{l_1}\right) + \left\{-2A_1r^{-3}a^{-1}(k_2-k_1)c_{44} - A_2l_2r^{-1}K_2\left(\frac{r}{l_2}\right)\right.$$

$$\left.-A_5\varepsilon_0l_1^{-1}K_1\left(\frac{r}{l_1}\right) - A_6\varepsilon_0l_1^{-1}\left[K_1\left(\frac{r}{l_1}\right) + 2l_1r^{-1}K_2\left(\frac{r}{l_1}\right)\right] + 2A_7r^{-3}a^{-1}\right\}\cos 2\theta, \tag{3.6c}$$

$$P_\theta = \left\{-2A_1r^{-3}a^{-1}(k_2-k_1)c_{44} - A_2\left[\frac{1}{2}K_1\left(\frac{r}{l_2}\right) + l_2r^{-1}K_2\left(\frac{r}{l_2}\right)\right]\right.$$

$$\left.-2A_6\varepsilon_0r^{-1}K_2 + 2A_7a^{-1}r^{-3}\right\}\sin 2\theta. \tag{3.6d}$$

Using the constitutive relations, we obtain the stress tensor τ_{ij} and E_{ij} that appear in the boundary conditions:

$$\tau_{rr} = -A_1c_{44}r^{-2}+A_3\alpha c_{44}r^{-2}-2A_5\varepsilon_0 l_1^{-1}(k_1-k_2)c_{44}r^{-1}K_1\left(\frac{r}{l_1}\right)$$
$$+\left\{-2A_1\alpha c_{44}r^{-2}-6A_4\alpha c_{44}r^{-4}+2A_6\varepsilon_0 l_1^{-1}(k_1-k_2)c_{44}\right.$$
$$\left.\times\left[6l_1r^{-2}K_2\left(\frac{r}{l_1}\right)+r^{-1}K_1\left(\frac{r}{l_1}\right)\right]+12A_7c_{44}(a^{-1}k_2+\varepsilon_0k_2-\varepsilon_0k_1)r^{-4}\right\}\cos 2\theta, \quad (3.7a)$$

$$\tau_{r\theta} = \left\{-A_1\alpha c_{44}r^{-2}-6A_4\alpha c_{44}r^{-4}+4A_6r^{-1}\varepsilon_0(k_2-k_1)c_{44}\left[l_1K_1\left(\frac{r}{l_1}\right)+3r^{-1}K_2\left(\frac{r}{l_1}\right)\right]\right.$$
$$\left.-12A_7\varepsilon_0(k_1-k_2)c_{44}r^{-4}\right\}\sin 2\theta, \quad (3.7b)$$

$$E_{rr} = A_1c_{44}[12l_0^2(k_2-k_1)r^{-2}-(k_1-k_2+2\alpha k_2)]r^{-2}\cos 2\theta-A_1d_{44}r^{-2}+\alpha A_3d_{44}r^{-2}$$
$$-2A_2al_0^2r^{-1}\left[K_1\left(\frac{r}{l_2}\right)+3r^{-1}l_2K_2\left(\frac{r}{l_2}\right)\right]-6A_4\alpha d_{44}r^{-4}\cos 2\theta$$
$$+A_5\left[(1+a\varepsilon_0)K_0\left(\frac{r}{l_1}\right)+2a\varepsilon_0 l_3^2l_1^{-1}r^{-1}K_1\left(\frac{r}{l_1}\right)\right]+A_6\left\{(1+a\varepsilon_0)K_2\left(\frac{r}{l_2}\right)+2a\varepsilon_0l_1^{-2}l_3^2\right.$$
$$\left.\times\left[r^{-1}l_1K_1\left(\frac{r}{l_1}\right)+6r^{-2}l_1^2K_2\left(\frac{r}{l_1}\right)\right]\right\}\cos 2\theta-12A_7[(1+a\varepsilon_0)l_0^2+a\varepsilon_0l_3^2]r^{-4}\cos 2\theta \quad (3.7c)$$

$$E_{(r\theta)} = \tfrac{1}{2}(E_{r\theta}+E_{\theta r}) = \left\{A_1[-\alpha d_{44}r^{-2}+12(b_{44}-k_2d_{44})c_{44}(k_2-k_1)a^{-1}r^{-4}]\right.$$
$$+A_2(b_{44}-k_2d_{44})\left[6l_2r^{-2}K_2\left(\frac{r}{l_2}\right)+r^{-1}K_1\left(\frac{r}{l_2}\right)+\frac{1}{2}l_2^{-1}K_2\left(\frac{r}{l_2}\right)\right]-6A_4\alpha d_{44}r^{-4}$$
$$+4A_6\varepsilon_0(b_{44}-k_1d_{44})\left[l_1^{-1}K_1\left(\frac{r}{l_1}\right)+3r^{-1}K_2\left(\frac{r}{l_1}\right)\right]r^{-1}$$
$$\left.+12A_7[\varepsilon_0d_{44}(k_2-k_1)-(b_{44}-k_2d_{44})a^{-1}]r^{-4}\right\}\sin 2\theta, \quad (3.7d)$$

$$E_{[r\theta]} = \frac{1}{2}(E_{r\theta}-E_{\theta r}) = \frac{1}{2}A_2b_{77}l_2^{-1}K_2\left(\frac{r}{l_2}\right). \quad (3.7e)$$

Substituting equations (3.7) in equations (3.2), equating coefficients of like functions of θ, we obtain a set of seven linear algebraic equations in the unknown constants $A_1, A_2, \ldots, A_7$.

$$g_{11}A_1+0+g_{13}A_3+0+g_{15}A_5+0+0 = -\frac{T}{2} \quad (3.8a)$$

$$g_{21}A_1+0+g_{23}A_3+0+g_{25}A_5+0+0 = -b_0 \quad (3.8b)$$

$$g_{31}A_1+0+0+g_{34}A_4+0+g_{36}A_6+g_{37}A_7 = -\frac{T}{2} \quad (3.8c)$$

$$g_{41}A_1+g_{42}A_2+0+g_{44}A_4+0+g_{46}A_6+g_{47}A_7 = 0 \tag{3.8d}$$

$$g_{51}A_1+0+0+g_{54}A_4+0+g_{56}A_6+g_{57}A_7 = \frac{T}{2} \tag{3.8e}$$

$$g_{61}A_1+g_{62}A_2+0+g_{64}A_4+0+g_{66}A_6+g_{67}A_7 = 0 \tag{3.8f}$$

$$A_2 b_{77} l_2^{-1} K_2\left(\frac{R}{l_2}\right) = 0. \tag{3.8g}$$

In which

$$
\begin{aligned}
g_{11} &= -c_{44}R^{-2} = -\alpha^{-1}g_{13} = \tfrac{1}{2}\alpha^{-1}g_{31} = \alpha^{-1}g_{51}\\
g_{15} &= -2\varepsilon_0 l_1^{-1}(k_1-k_2)c_{44}R^{-1}K_1\left(\frac{R}{l_1}\right)\\
g_{21} &= -d_{44}R^{-2} = -\alpha^{-1}g_{23}\\
g_{25} &= \left[(1+\eta^{-1})K_0\left(\frac{R}{l_1}\right)+2\eta^{-1}l_3^2 l_1^{-1}R^{-1}K_1\left(\frac{R}{l_1}\right)\right]\\
g_{34} &= -6\alpha c_{44}R^{-4} = g_{54}\\
g_{36} &= -2k_2^{-1}\eta^{-1}(l_0^2-l_3^2)\left[6R^{-2}K_2\left(\frac{R}{l_1}\right)+l_1^{-1}R^{-1}K_1\left(\frac{R}{l_1}\right)\right]\\
g_{37} &= -12\eta^{-1}k_2^{-1}(l_0^2-l_3^2)R^{-4} = g_{57}\\
g_{41} &= 12(k_2-k_1)c_{44}l_0^2R^{-4}-[(k_1-k_2)+2\alpha k_2]c_{44}R^{-2}\\
g_{42} &= 2al_0^2R^{-1}(K_1+3l_2R^{-1}K_2)\\
g_{46} &= (1+\eta^{-1})K_2\left(\frac{R}{l_1}\right)+2\eta^{-1}l_3^2\left[6R^{-2}K_2\left(\frac{R}{l_1}\right)+l_1^{-1}R^{-1}K_1\left(\frac{R}{l_1}\right)\right]\\
g_{47} &= -12l_0^2R^{-4}-12\eta^{-1}(l_0^2-l_3^2)R^{-4} = g_{67}\\
g_{56} &= -47^{-1}k_2^{-1}(l_0^2-l_3^2)\left[3R^{-2}K_2\left(\frac{R}{l_1}\right)+l_1^{-1}R^{-1}K_1\left(\frac{R}{l_1}\right)\right]\\
g_{61} &= -\alpha k_2c_{44}R^{-2}+12(k_2-k_1)c_{44}l_0^2R^{-4}\\
g_{62} &= al_0^2\left[6l_2R^{-2}K_2\left(\frac{R}{l_2}\right)+R^{-1}K_1\left(\frac{R}{l_2}\right)+\frac{1}{2}l_2^{-1}K_2\left(\frac{R}{l_2}\right)\right]\\
g_{64} &= -6\alpha d_{44}R^{-4} = g_{44}\\
g_{66} &= 4\eta^{-1}l_3^2\left[3R^{-2}K_2\left(\frac{R}{l_1}\right)+l_1^{-1}R^{-1}K_1\left(\frac{R}{l_1}\right)\right]
\end{aligned}
\tag{3.9}
$$

where

$$
\begin{aligned}
l_0^2 &= a^{-1}(b_{44}-k_2 d_{44}),\\
l_1^2 &= a^{-1}(1+\eta)^{-1}\{(b_{12}+2b_{44})-k_1(d_{12}+2d_{44})\},\\
l_2^2 &= a^{-1}(b_{44}+b_{77}-k_2 d_{44}),\\
l_3^2 &= a^{-1}(b_{44}-k_1 d_{44}),\\
\eta^{-1} &= a\varepsilon_0.
\end{aligned}
\tag{3.10}
$$

The solution of the seven equations is

$$A_1 = \frac{TR^2}{\alpha c_{44}}(1-MN_1), \tag{3.11a}$$

$$A_2 = 0, \tag{3.11b}$$

$$A_3 = -\frac{TR^2}{2\alpha^2 c_{44}}[\alpha-2(1-MN_1)]+\frac{R^2 N_1}{2\alpha c_{44} N_0}\left(T-\frac{2b}{k_2}\right), \tag{3.11c}$$

$$
\begin{aligned}
A_4 = {} & -\frac{TR^4}{4\alpha c_{44}}-\frac{TR^4 MN_1}{6\alpha c_{44}}+\frac{TR^4}{3\alpha c_{44}}\eta^{-1}(l_0^2-l_3^2)\\
& \times\left[\frac{1}{8l_0^2}+\frac{MN_3}{4l_0^2}+\frac{3M}{R^2}K_2\left(\frac{R}{l_1}\right)-\frac{(k_2-k_1)}{4\alpha K_2 l_0^2}(1+12l_0^2R^{-2})(1-MN_1)\right],
\end{aligned}
\tag{3.11d}
$$

$$A_5 = \frac{k_2 T-2b_0}{2N_0}, \tag{3.11e}$$

$$A_6 = -k_2 MT, \tag{3.11f}$$

$$A_7 = -\frac{k_2 TR^4}{48l_0^2}(1+2MN_3)+\frac{(k_2-k_1)TR^4}{24\alpha l_0^2}(1-MN_1)(1+12l_0^2R^{-2}), \tag{3.11g}$$

where

$$N_0 = (1+\eta^{-1})K_0\left(\frac{R}{l_1}\right)+2\eta^{-1}l_0^2 l_1^{-1}R^{-1}K_1\left(\frac{R}{l_1}\right), \tag{3.12a}$$

$$N_1 = 2\eta^{-1}(l_0^2-l_3^2)l_1^{-1}R^{-1}K_1\left(\frac{R}{l_1}\right), \tag{3.12b}$$

$$N_2 = (1+\eta^{-1})K_2\left(\frac{R}{l_1}\right)-2\eta^{-1}l_3^2 l_1^{-1}R^{-1}K_1\left(\frac{R}{l_1}\right), \tag{3.12c}$$

$$N_3 = (1+\eta^{-1})K_2\left(\frac{R}{l_1}\right)+12\eta^{-1}l_0^2R^{-2}K_2\left(\frac{R}{l_1}\right)+2\eta^{-1}l_0^2 l_1^{-1}R^{-1}K_1\left(\frac{R}{l_1}\right), \tag{3.12d}$$

$$M = \frac{k_1-k_2+\alpha k_2}{(k_1-k_2+\alpha k_2)N_1-\alpha N_2 k_2}. \tag{3.12e}$$

Substituting equations (3.6) in the constitutive relations, we get the circumferential component of stress:

$$\tau_{\theta\theta} = A_1c_{44}r^{-2} - A_3\alpha c_{44}r^{-2} + 6A_4\alpha c_{44}r^{-4}\cos 2\theta + 2A_5\varepsilon_0 l_1^{-1}c_{44}(k_1-k_2)$$
$$\times\left[r^{-1}K_1\left(\frac{r}{l_1}\right) + l_1^{-1}K_0\left(\frac{r}{l_1}\right)\right] + 2A_6\varepsilon_0 c_{44}(k_1-k_2)$$
$$\times\left[r^{-1}\left(6r^{-1}K_2\left(\frac{r}{l_1}\right) + l_1^{-1}K_1\left(\frac{r}{l_1}\right)\right) + l_1^{-2}K_2\left(\frac{r}{l_1}\right)\right]\cos 2\theta$$
$$+12A_7k_2^{-1}\eta^{-1}(l_0^2-l_3^2)r^{-4}\cos 2\theta + \frac{T}{2}(1-\cos 2\theta). \tag{3.13}$$

The maximum value of $\tau_{\theta\theta}$ at the surface of the hole occurs at $\theta = \pm\pi/2$. We find

$$F_c = \frac{[\tau_{\theta\theta}]_{\max}}{T} = 3 + \frac{b_0k_2^{-1}}{TN_0}\left\{N_1 - \frac{2\eta^{-1}(l_0^2-l_3^2)}{l_1^2}\left[K_1\left(\frac{R}{l_1}\right) + \frac{l_1}{R}K_1\left(\frac{R}{l_1}\right)\right]\right.$$
$$+MN_1 - \frac{\eta^{-1}}{2}(l_0^2-l_3^2)\left[3MR^{-2}K_2\left(\frac{R}{l_1}\right) + \frac{1}{8l_0^2}(1+2MN_3)\right.$$
$$\left.-\frac{k_2-k_1}{4\alpha l_0^2k_2}(1+12l_0^2R^{-2})(1-MN_1)\right] + \frac{\eta^{-1}(l_0^2-l_3^2)}{N_0l_1^2}\left[K_0\left(\frac{R}{l_1}\right) + \frac{l_1}{R}K_1\left(\frac{R}{l_1}\right)\right]$$
$$+\frac{2M\eta^{-1}(l_0^2-l_3^2)}{l_1^2}\left\{K_2\left(\frac{R}{l_1}\right) + \frac{l_1}{R}\left[6\frac{l_1}{R}K_2\left(\frac{R}{l_1}\right) + K_1\left(\frac{R}{l_1}\right)\right]\right\}$$
$$\left.+\frac{\eta^{-1}(l_0^2-l_3^2)}{l_0^2}\left[\frac{1}{4} + \frac{1}{2}MN_3 - \frac{(k_2-k_1)}{2\alpha k_2}(1+12l_0^2R^{-2})(1-MN_1)\right].\right. \tag{3.14}$$

The quantity F_c, so defined, is the stress concentration factor.

4. STRESS-CONCENTRATION FACTOR

The result of the previous section shows that the stress concentration factor F_c depends upon the radius of the hole, three length properties of the material l_0, l_1 and l_3. Poisson's ratio ν, electromechanical coupling factors k_1 and k_2, and the reciprocal dielectric susceptibility η^{-1}. The present continuum theory is concerned only with macroscopic cylindrical hole and since l_0, l_1 and l_3 are of the order of magnitude of the interatomic distances so that, in the domain of validity of the continuum hypothesis, R/l_0, R/l_1 and R/l_3 are large numbers. After making use of the asymptotic representation $(\pi/2\chi)^{\frac{1}{2}}\,e^{-\chi}$ for the Bessel function $K_n(\chi)$ [3], we get stress concentration factor

$$F_c = \frac{[\tau_{\theta\theta}]_{r=R,\theta=\pm\pi/2}}{T} = 3 + \frac{f_0}{T} + f_1 \tag{4.1a}$$

where

$$f_0 = -\frac{2b_0(c_{44}d_{11} - c_{11}d_{44})}{b_{11}c_{11} - d_{11}^2}, \tag{4.1b}$$

and

$$f_1 = -\frac{d_{44}(c_{44}d_{11}-c_{11}d_{44})}{c_{44}(b_{11}c_{11}-d_{11}^2)}. \tag{4.1c}$$

Thus, if the constants b_0, d_{12} and d_{44}, the coefficients associated with terms involving the product of polarization gradient and strain in the energy density are neglected, the second and third terms in equation (4.1) are zero and the stress-concentration factor reduces to the usual value 3 [4]. The second term in equation (4.1) is the concomitant stress arising from the surface energy at free surface given by Mindlin [1]. According to the values given by Askar *et al.* [5] the coefficient f_0 is positive and of the same order of magnitude as c_{11}, the elastic stiffness of the material. This term represents the interaction of the applied stress and surface energy. By the requirement that the energy density W must be positive definite, it can be shown that

$$\begin{aligned} b_{11}c_{11}-d_{11}^2 &> 0 \\ c_{44}d_{11}-c_{11}d_{44} &> 0 \end{aligned} \tag{4.2}$$

and

$$d_{44} < 0.$$

Thus the third term is always positive. Coupling the solutions for three simple problems of homogeneous deformation, viz. simple tension, hydrostatic pressure, and shear with the conditions for positive definiteness of the potential energy density, the appropriate range for f_1 is $0 < f_1 < \frac{1}{12}$. Within this range, the classical stress concentration factor is about 10 per cent less that that given by (4.1), even though the surface energy effect is neglected. It appears that the fracture strength and the onset of static yielding on some dielectric materials in the presence of stress concentration may occur at lower loads than might be expected on the basis of stress concentration factors calculated from the classical theory of elasticity.

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REFERENCES

[1] R. D. MINDLIN, Polarization gradient in elastic dielectrics. *Int. J. Solids Struct.* **4**, 637–642 (1968).
[2] J. SCHWARTZ, Solutions of the equations of equilibrium of elastic dielectrics. *Int. J. Solids Struct.* **5**, 1209–1211 (1969).
[3] B. PEIRCE, *A Short Table of Integrals*, p. 94. Ginn (1956).
[4] S. TIMOSHENKO and J. N. GOODIER, *Theory of Elasticity*, 2nd edition, p. 78. McGraw-Hill (1951).
[5] A. ASKAR, P. C. Y. LEE and A. S. CAKMAK, The Effect of Surface Curvature and Discontinuity on the Surface Energy Density and Other Induced Fields in Elastic Dielectrics with Polarization Gradient, Research Report No. 70-Mech. 1, Department of Civil and Geological Engineering, Princeton University.

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Абстракт—Даётся решение задачи цилиндрического отверстця в полю продольного растяжения, в рамках линейной теории упругих диэлектриков. В этих диэлектриках, потенцияльная энергия плотности деформации и поляризации зависит как от градиента поляризации, так и от самой деформации и поляризации.

Даётся коэффициента концентрации напряжений на поверхнрсти цилиндрического отверстия.